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# DLCQ of Fivebranes, Large $N$ Screening, and $L^2$ Harmonic Forms on Calabi Manifolds

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## ABSTRACT

We find one explicit  $L^2$  harmonic form for every Calabi manifold. Calabi manifolds are known to arise in low energy dynamics of solitons in Yang-Mills theories, and the  $L^2$  harmonic form corresponds to the supersymmetric ground state. As the normalizable ground state of a single  $U(N)$  instanton, it is related to the bound state of a single D0 to multiple coincident D4's in the non-commutative setting, or equivalently a unit Kaluza-Klein mode in DLCQ of fivebrane worldvolume theory. As the ground state of nonabelian massless monopoles realized around a monopole-“anti”-monopole pair, it shows how the long range force between the pair is screened in a manner reminiscent of large  $N$  behavior of quark-anti-quark potential found in AdS/CFT correspondence.

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# 1 Introduction

Several recent research issues have come together to the study of the supersymmetric quantum dynamics on a family of hyper-Kähler manifolds, one for each  $4n + 4$  space with nonnegative integer  $n$ . These manifolds, the so-called Calabi manifolds [1], arise naturally as the moduli spaces of certain configurations of magnetic monopoles [2] or a single instanton on a noncommutative space [3].

The Calabi manifold is perhaps the simplest class of nontrivial hyper-Kähler manifolds. The Calabi manifold of dimension  $4n + 4$  is  $T^*CP^{n+1}$ , the cotangent bundles of the complex projective space  $CP^{n+1}$ , equipped with a hyper-Kähler metric of cohomogeneity one and the triholomorphic isometry  $SU(n+2)$  [4]. There is one scaling coordinate for the size of the orbit and the generic orbit of the isometry is the coset space  $SU(n+2)/U(n)$  of dimension  $4n + 3$ , which collapses to a  $CP^{n+1}$  bolt in the minimal scaling.

One form of the metric is

$$C_{AB}d\vec{x}_A \cdot d\vec{x}_B + (C^{-1})_{AB}(d\psi_A + \vec{\omega}_{AC} \cdot d\vec{x}_C)(d\psi_B + \vec{\omega}_{BD} \cdot d\vec{x}_D). \quad (1)$$

The coordinate  $\psi_A$  has the period  $4\pi$  and the symmetric matrix  $C_{AB}$  has the form,

$$C_{AB} = \left( \frac{\delta_{AB}}{|\vec{x}_A|} + \frac{1}{|\sum_E \vec{x}_E - \vec{R}|} \right). \quad (2)$$

while the vector potentials  $\vec{\omega}_{AB}$  solves the equations

$$\vec{\nabla}_D C_{AB} = \vec{\nabla}_D \times \vec{\omega}_{AB}. \quad (3)$$

This form of metric naturally arises as a hyper-Kähler reduction of flat  $R^{4n+8}$  [5]. The hyper-Kähler reduction preserves some of the isometries: More specifically, there is an  $SU(n+2)$  isometry which preserves not only the metric but also all three kähler forms, while an additional  $U(1)$  rotating  $\vec{x}_A$ 's orthogonally to  $\vec{R}$ , preserves the metric and one kähler form. Such isometries are called triholomorphic and holomorphic, respectively.

Recently, there has been an interesting progress in the study of the Calabi manifold. Ref. [6] presented an alternate form of the metric, which was then used to find the explicit expression of an  $L^2$  harmonic form for eight and twelve dimensional manifolds. Here, we use this new form for the metric to find the explicit expression of one normalizable harmonic form on the Calabi manifold of arbitrary dimensions.

The  $L^2$  harmonic form is anticipated to be unique on each Calabi manifold [7], and therefore must be self-dual or anti-self-dual middle form. We indeed find such an  $L^2$  harmonic form.

We will then explore the physical implications in two physical settings; in the context of (2,0) theories in six dimensions [8, 9, 10] and in the context of the large  $n$  dynamics of monopole and anti-monopole in partially broken gauge theories. Recall that a normalizable harmonic form, if it exists, corresponds to a normalizable ground state wave function of the Hamiltonian with eight supersymmetries. This middle form has several physical meanings. The four dimensional case with  $n = 0$  is the so-called Eguchi-Hanson metric [11]. The normalizable middle form in this space has been found and studied in many places [12, 3, 13].

Initially, Calabi discovered this family of metrics as a study of the hyper-Kähler space. It has also appeared in the study of the moduli space of magnetic monopoles and instantons. The simplest example is the moduli space of a single instanton in  $U(n+2)$  theory [3]. Its moduli space in the center of mass frame is  $4(n+1)$  dimensional hyper-Kähler space with one scale parameter and the principal orbit  $SU(n+2)/U(n)$ . This space is singular at the zero scale size, which can be blown up to a finite bolt, making the space nonsingular. This has been done in the ADHM formalism. This blow-up of the instanton moduli space can be achieved physically by going to the non-commutative space, which is equivalent to introducing a FI term on the ADHM formalism.

Recall that  $U(m)$  type (2,0) theories arise as low energy dynamics of  $m$  coincident fivebranes [10]. Alternatively, it also arises as the strong coupling limit of the five dimensional  $U(m)$  Yang-Mills theories with 16 supersymmetries. Not much is known about these theories beyond their existence and somewhat tentative statements about their anomaly structure [14, 15, 16]. The only explicit approach to these theories, known to date, is the DLCQ limit thereof [17]. Because instanton solitons of five-dimensional Yang-Mills theory carries the Kaluza-Klein momentum modes along the extra circle, DLCQ of the (2,0) theories have partons that are really instanton solitons. This approach to the (2,0) theories is thus naturally tied to instanton dynamics of five dimensional Yang-Mills theories. An early study of the relationship was conducted by Aharony et.al. [18], who tried to extract spectrum of chiral primaries of (2,0) theories from cohomology counting on the instanton moduli space. While they found numerous nontrivial cohomology generators, only one of them is associated with free

center of mass degrees of freedom and may correspond to  $L^2$  harmonic form on Calabi space. Our findings here show that indeed one such  $L^2$  harmonic form exists for any number of coincident fivebranes. Extrapolation of this correspondence to arbitrary number  $k$  of instantons and arbitrary  $m$ , gives us a conjecture that there should be a unique  $L^2$  harmonic form for each and every pair  $m$  and  $k$ . Our finding in this paper supports the conjecture by explicitly constructing such harmonic forms for all  $n$  and  $k = 1$ .<sup>1</sup>

Calabi manifold and the  $L^2$  harmonic form on it find another application in the moduli space dynamics of certain non-Abelian monopoles [19]. Consider  $N = 4$   $SU(n+4)$  theory broken to  $U(1)^{n+3}$ , which contains  $n+3$  distinct types of fundamental monopoles. The moduli space of these  $n+3$  distinct monopoles is explicitly known [20]. The moduli space dimension is  $4(n+3)$  and each four degrees correspond to the position and phase of each fundamental monopole. When the gauge symmetry is partially restored to  $U(1) \times SU(n+2) \times U(1)$  so that only two of them at the end of the Dynkin diagram remain massive, the net magnetic charge is orthogonal to unbroken  $SU(n+2)$  generators. In this limit the moduli space remains still sensible and still is of  $4(n+3)$  dimensions [2].

This moduli space turns out to have two scale parameters, one for the relative distance between two massive monopoles and one for the scale parameter for the massless monopole clouds surrounding two massive monopoles. This latter corresponds to the sum of the relative distances between pairs of interacting monopoles, some of which is now massless can no longer be regarded as isolated solitons. In such a massless limit, only the two scales are invariant under global gauge rotations and spatial rotations. All the rest are associated with  $U(1) \times SU(n+2) \times U(1)$  or  $SO(3)$  rotation of the solution. The relative moduli space of dimension  $4(n+2)$  has  $U(n+2)$  triholomorphic isometry and  $SU(2)$  rotational symmetry which is just holomorphic with respect to one of three complex structure. The particle dynamics on such a moduli space has been studied extensively in a recent paper [21].

When we take a further limit of two massive particles becoming infinitely massive so that their relative distance is fixed, or after the hyper-Kähler quotient of the Taubian-Calabi metric with the  $U(1)$  triholomorphic isometry, the resulting  $4(n+1)$  dimensional metric is the Calabi metric which we study. Since magnetic monopoles

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<sup>1</sup>In the paper [13] by one of us, the moduli space of two  $U(1)$  instantons is shown to be Eguchi-Hanson which possesses as a unique normalizable middle form in its moduli space. This state was interpreted as the mode for two momenta, giving an independent check for the  $k = 2, m = 1$  case.

in the  $N = 4$  supersymmetric four dimensional Yang-Mills Higgs theories have half supersymmetry, the moduli space metric has the eight supersymmetries.

One of the key achievement in the AdS-CFT correspondence between the classical supergravity theory on the  $AdS_5 \times S^5$  and  $N = 4$  supersymmetric Yang-Mills theory is the calculation of the potentials between quarks in the large  $e^2 N$  limit, keeping  $e^2$  small [22, 23]. The non-analytical expression  $\sqrt{e^2 N}$  has been found as the coefficient. There has been several works in the field theory to reproduce this highly nontrivial result [24]. The present work is partially motivated by this result. Here we study the potential in the large  $N$  limit. As the non-perturbative reduction of the attractive quarks is happening in the low energy when the overall coefficient is quite small, one may expect that the low energy dynamics we study here may capture the right physics. As we will see our result is encouraging but not identical to the AdS-CFT calculation. One may regard our work as a first attempt to understand the AdS-CFT result in our direction.

In Section 2, we consider DLCQ of fivebranes and show how Calabi manifold appears naturally in that context. By turning on non-commutativity, we may reduce the quantum mechanics to quantum mechanics on the resolved instanton moduli space of  $U(m)$  theory. The latter is nothing but the Calabi manifold. In Section 3, we study the metric of the Calabi space introduced recently. There we find the explicit expression for the normalizable middle forms for each  $n$ . In Section 4, we interpret the harmonic form as corresponding to unit Kaluza-Klein mode associated with free center of mass motion of coincident fivebranes. In Section 5, we study an unrelated application of the harmonic form. We consider static potential between a pair of non-Abelian monopoles which can be regarded as a monopole-anti-monopole pair. We find an screening effect at large  $n$ , reminiscent of a similar effect observed in AdS/CFT setting, even though we are working at the level of low energy dynamics only. In Section 6, we conclude with a summary.

## 2 DLCQ of Fivebrane Theory and Instanton

Instanton of five dimensional Yang-Mills theories plays the role of partons when one considers DLCQ limit of coincident fivebranes. For each instanton number, one is probing dynamics with fixed total Kaluza-Klein momentum sector, and this correspondence predicts existence of certain bound state on DLCQ description. In par-

ticular, given any number of fivebranes, there is always a free center of mass part of the world-volume dynamics, given by a single tensor multiplet. These degrees of freedom must manifest themselves as normalizable supersymmetric ground states. As we explain below, the  $L^2$  harmonic form we found provide exactly such state, and gives us a consistency check on the DLCQ prescription.

The actual DLCQ quantum mechanics is some Yang-Mills type quantum mechanics with 8 supercharges, Higgs branch of which corresponds to instanton moduli space while Coulomb branch corresponds to freed D0 branes in spacetime [17]. Because we are dealing with quantum mechanics, a clean separation between branches is not possible, and in general one must consider the full quantum mechanics in order to understand dynamics of fivebrane world-volume [25, 26]. On the other hand, problem gets quite simplified when non-commutativity is turned on. This removes the Coulomb branch altogether, and leaves behind a resolved version of the instanton moduli space as the Higgs branch.<sup>2</sup> With this, the system may be reduced to sigma model onto the resolved instanton moduli space, which we will presently show to be exactly the Calabi manifold. Here we will use the result of Ref.[3], and present the moduli space of non-commutative instanton defined on  $R^3 \times S^1$  and then derive Calabi manifold as instanton moduli space on  $R^4$  by taking the decompactification limit..

Consider a periodic instanton on  $R^3 \times S^1$ , of non-commutative  $U(n+2)$  theory in  $4+1$  dimensions. Such a periodic instanton is known to consist of  $n+2$  partons which are BPS monopoles [28]. Using this realization, the instanton moduli space has been computed. The monopoles involved are all distinct, and the metric of low energy dynamics is toric with  $n+2$   $U(1)$  triholomorphic isometries. Only interaction between distinct monopoles is due to exchange of purely  $U(1)$  massless vector multiplets, which comes in a scale invariant form. This uniquely fixes the moduli space metric.

Separating out flat  $R^3 \times S^1$  that corresponds to pure translational degrees of freedom, the moduli space is a  $4n+4$  dimensional hyper-Kähler space. Let  $l$  be radius of  $S^1$  and  $e$  the five dimensional Yang-Mills coupling. The metric can be written as

$$\mathcal{G} = \frac{4\pi^2 l}{e^2} \left( C_{AB} d\vec{x}_A \cdot d\vec{x}_B + (C^{-1})_{AB} (d\psi_A + \vec{\omega}_{AC} \cdot d\vec{x}_C) (d\psi_B + \vec{\omega}_{BD} \cdot d\vec{x}_D) \right). \quad (4)$$

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<sup>2</sup>Refs. [27] addressed the question of whether a single D0 binds to a single D4 without any such deformation. Generalization of this approach to arbitrary number of D4 is significantly more difficult and remains an open problem.

$\psi_A$  are periodic in  $4\pi$ , and the symmetric matrix  $C_{AB}$  has the form,

$$C_{AB} = \left( \mu_{AB} + \frac{\delta_{AB}}{|\vec{x}_A|} + \frac{1}{|\sum_{E=1}^{n+1} \vec{x}_E - 2\pi\vec{\zeta}/l|} \right). \quad (5)$$

The hyper-Kähler property requires the vector potentials  $\vec{\omega}_{AB}$  to be related to  $C_{AB}$  by

$$\vec{\nabla}_D C_{AB} = \vec{\nabla}_D \times \vec{\omega}_{AB}. \quad (6)$$

The constant matrix  $\mu$  is proportional to the reduced mass matrix of the partons, and is determined uniquely by the Wilson line along  $S^1$ . In particular, it is non-degenerate in the maximally broken phase, and vanishes identically when the gauge symmetry is completely restored. Three vectors  $\vec{x}$  are relative positions of the partons in  $R^3$ .

Non-commutativity is encoded in the vector  $\vec{\zeta}$ , which is the anti-self-dual part of the commutator,  $\theta_{\mu\nu} = i[x_\mu, x_\nu]$ , which defines the non-commutativity of the  $R^3 \times S^1$ . The anti-self-dual part is effectively a vector under  $SO(3)$  of  $R^3$ .

Let us take the limit where the  $U(n+2)$  gauge symmetry is restored by turning off Wilson line. In effect, we set  $\mu = 0$ .

$$C_{AB} = \left( \frac{\delta_{AB}}{|\vec{x}_A|} + \frac{1}{|\sum_{E=1}^{n+1} \vec{x}_E - 2\pi\vec{\zeta}/l|} \right). \quad (7)$$

Note that, apart from an overall multiplicative scale, corresponding metric is exactly that of Eq. (1). Thus the instanton moduli space in the limit of vanishing Wilson line is a Calabi manifold.

There is a further limit we can take while maintaining Calabi manifold as the moduli space. We may choose to go back to  $R^4$  by sending  $l$  to infinity. This may seem like a singular limit, given that  $\zeta/l$  would vanish. However, such a limit should describe a resolved instanton moduli space which is smooth. The resolution of the apparent conflict lies in the fact that there is an overall  $l$  multiplying the metric, so in order to reach a finite limit, one must rescale the collective coordinates by  $\vec{x} \rightarrow \vec{x}/l$ , upon which the moduli space remains Calabi manifold with  $\vec{R} = 2\pi\vec{\zeta}$ .

Thus, an instanton on non-commutative  $R^3 \times S^1$  or on non-commutative  $R^4$  is characterized by the common, nontrivial part of the moduli space given by the Calabi manifold. The only difference lies in the flat, center of mass part of the moduli space,  $R^3 \times S^1$  and  $R^4$ , respectively. In next section, we will introduce a new coordinate system, where quantization procedure proves a bit easier, and find a normalizable, bound state at zero excitation energy. *Thus we effectively will have shown the existence of the Kaluza-Klein mode associated with the free part of the fivebrane theory for*

any number of fivebranes. Unlike the case of the matrix theory in the bulk [29, 30], this is a rather nontrivial test even for the unit Kaluza-Klein mode, since the parton itself carries many internal degrees of freedom which must be quantized.

### 3 $L^2$ Harmonic Form on Calabi Manifolds

The above metric, although quite simple, is somewhat unwieldy for two reasons: neither the complex structures nor the  $SU(n+2)$  isometry is easy to see in this form. An alternate form was recently found, which take advantage of the principal orbits [6]. In this second version, the metric is written in terms of a single radial coordinate  $\rho$  and a 1-form frame on  $S^{4n+3}$ ,  $\sigma_a$ ,  $\bar{\sigma}_a$ ,  $\Sigma_a$ ,  $\bar{\Sigma}_a$ ,  $\nu$ ,  $\bar{\nu}$ , and  $\lambda$ ;

$$h(\rho)^2 d\rho^2 + a(\rho)^2 \sum_{a=1}^n |\sigma_a|^2 + b(\rho)^2 \sum_{a=1}^n |\Sigma_a|^2 + c(\rho)^2 |\nu|^2 + f(\rho)^2 \lambda^2, \quad (8)$$

with appropriate functions

$$a(\rho)^2 = (\rho^2 - 1)/2, \quad b(\rho)^2 = (\rho^2 + 1)/2, \quad c(\rho)^2 = \rho^2, \quad (9)$$

and

$$h(\rho)^2 = \rho^4/(\rho^4 - 1), \quad f(\rho)^2 = \rho^2(1 - 1/\rho^4)/4. \quad (10)$$

This metric is equivalent to the above up to an overall rescaling of the metric, upon relating the radial coordinate  $\rho$  with  $\vec{x}_A$  by

$$\rho^2 = \left( \left( \sum_A |\vec{x}_A| \right) + \left| \sum_A \vec{x}_A - \vec{R} \right| \right) / R. \quad (11)$$

#### 3.1 Invariant Form on Calabi Manifold

The manifold comes with a triholomorphic  $SU(n+2)$  isometry, and also a  $U(1)_\lambda$  isometry whose dual is  $f^2\lambda$ . Under the  $U(1)_\lambda$  isometry, whose Killing vector is dual to  $f^2\lambda$ , the complex 1-forms are charged as,

$$\begin{aligned} \sigma_a &\rightarrow 1/2, \\ \Sigma_a &\rightarrow -1/2, \\ \nu &\rightarrow 1. \end{aligned} \quad (12)$$

Alternatively, we may view  $S^{4n+3}$  as a deformed version of coset  $SU(n+2)/U(n)$  where  $U(n)$  acts on the right. In this latter viewpoint, the above 1-forms are part



of left-invariant 1-form basis of  $SU(n+2)$  group manifold, transforming as adjoint under the right action of  $SU(n+2)$ . In particular, the above 1-forms are grouped into

$$\begin{aligned}\sigma_a &\rightarrow [n]_1, \\ \Sigma_a &\rightarrow [n]_1, \\ \nu &\rightarrow [1]_0,\end{aligned}\tag{13}$$

under the  $U(n)$  as the subgroup of right  $SU(n+2)$ .

For convenience, we will define the complex orthonormal basis

$$\begin{aligned}s &\equiv h d\rho + i f \lambda, \\ p &\equiv c \nu, \\ \xi_k &\equiv a \sigma_k, \\ \zeta_k &\equiv b \Sigma_k,\end{aligned}\tag{14}$$

with the corresponding conjugated ones denoted by an overbar.

The manifold is hyper-Kähler, which means that there are three covariantly constant Kähler forms,

$$\begin{aligned}K^{(1)} &= s\bar{s} + p\bar{p} + \sum_k \xi_k \bar{\xi}_k - \sum_k \zeta_k \bar{\zeta}_k, \\ K^{(2)} + iK^{(3)} &= sp + i \sum_k \xi_k \bar{\zeta}_k.\end{aligned}\tag{15}$$

The associated complex structures  $J^{(a)}$  form an  $SU(2)_J$  algebra, which is an R-symmetry of the supersymmetric quantum mechanics onto the Calabi manifold. Under this  $SU(2)_J$ , the 1-forms are split into  $(2n+2)$  doublets, which are

$$\begin{pmatrix} \xi_k \\ i\zeta_k \end{pmatrix},\tag{16}$$

and

$$\begin{pmatrix} i\bar{\zeta}_k \\ \bar{\xi}_k \end{pmatrix},\tag{17}$$

for each  $k = 1, \dots, n$ , and also

$$\begin{pmatrix} s \\ \bar{p} \end{pmatrix},\tag{18}$$

and

$$\begin{pmatrix} -p \\ \bar{s} \end{pmatrix}.\tag{19}$$

We will be looking for a normalizable wavefunction of the complex  $N = 4$  supersymmetric quantum mechanics. In other words, we will search for  $L^2$  harmonic forms on the Calabi manifold. Such a harmonic form is anticipated to be unique on Calabi manifold [7], and we will assume that this is the case throughout the paper.

The uniqueness imposes several constraint immediately. First, it should be either self-dual or anti-self-dual form, which also implies that it is a middle-dimensional form. Second, the uniqueness implies that it is a singlet under  $U(n)$  and  $U(1)_\lambda$  and also under the  $SU(2)_J$  R-symmetry. The latter, in particular, proves to be very constraining. It implies that the middle form is of the form,

$$\Psi = \sum F_a(\rho)\Omega_a, \quad (20)$$

where middle forms  $\Omega_a$ 's are constructed out of the orthonormal basis above and invariant under  $U(n) \times U(1)_\lambda \times SU(2)_J$ .

After playing with the basis, one can see that any such invariant middle  $\Omega_a$  can be constructed by multiplying the following 2-forms and 4-forms,

$$\begin{aligned} A &\equiv s\bar{s} - p\bar{p}, \\ B &\equiv \sum_k \xi_k \bar{\xi}_k + \zeta_k \bar{\zeta}_k, \\ C &\equiv -2isp \sum_k \zeta_k \bar{\xi}_k - 2i\bar{s}\bar{p} \sum_k \xi_k \bar{\zeta}_k + (s\bar{s} + p\bar{p}) \sum_k (\xi_k \bar{\xi}_k - \zeta_k \bar{\zeta}_k), \\ D &\equiv -\sum_k \xi_k \bar{\zeta}_k \sum_k \zeta_k \bar{\xi}_k + \sum_k \xi_k \bar{\xi}_k \sum_k \zeta_k \bar{\zeta}_k. \end{aligned} \quad (21)$$

It should be immediately clear that they are invariant under  $U(n)$  and also under  $U(1)_\lambda$ . Invariance under  $SU(2)_J$  needs a little bit more of scrutiny.  $A$  and  $B$  are constructed using the usual invariant bilinear form of  $SU(2)_J$  while  $C$  and  $D$  are constructed by first forming a pair of  $SU(2)_J$  triplets that are also invariant under  $SU(n)$ , and then multiplying two such to form a singlet. We could have used the totally antisymmetric tensor of  $SU(n)$  to construct invariant forms, but when we require  $SU(2)_J$  invariance of the final middle form, they can be rewritten in terms of the above four. In  $4n + 4$  dimensions, invariant middle forms are,

$$\begin{aligned} D^l B^{n-2l+1}, \\ D^l A B^{n-2l}, \\ D^l A^2 B^{n-2l-1}, \\ D^l C B^{2-2l-1}, \end{aligned} \quad (22)$$

with  $l = 0, 1, 2, \dots$  etc. There is a crucial identity among the first set that goes as

$$0 = \sum_{l=0}^{[n/2]} (-1)^l \binom{n-l+1}{l} D^l B^{n-2l+1}. \quad (23)$$

which comes about due to the antisymmetric nature of the wedge product and the fact that we are forming singlets by multiplying  $n$  doublets  $n+1$  times.<sup>3</sup>

### 3.2 $L^2$ Harmonic Form

In principle, the self-dual middle form  $\Psi$  is harmonic if and only if it is closed,

$$d\Psi = 0. \quad (24)$$

On the other hand, invariance under  $SU(2)_J$  implies that the middle form is of type  $(n+1, n+1)$  with respect to any Hodge decomposition. This implies that  $\Psi$  is closed if and only if it is closed under holomorphic exterior derivative,

$$\partial\Psi = 0. \quad (25)$$

We will solve this equation. For this, we need the exterior algebra for the left-invariant 1-forms on  $SU(n+2)$  which is summarized in Appendix A. Since the middle form  $\Psi$  is entirely constructed in terms of the forms  $A$ ,  $B$ ,  $C$  and  $D$ , it is convenient to have their derivatives expressed in terms of them. From the exterior algebra given in Appendix A, we find

$$\begin{aligned} \partial A &= \frac{4f}{c^2} sA - \frac{1}{2fc^2} sB - \frac{1}{2f} E, \\ \partial B &= \frac{1}{2f} sB + \frac{1}{2fc^2} E, \\ \partial C &= -\frac{1}{2f} sB^2 - \frac{3}{2fc^2} sAB + \frac{2}{f} sD - \frac{3}{2f} EA - \frac{1}{2fc^2} EB, \\ \partial D &= \frac{1}{f} sD, \end{aligned} \quad (26)$$

where we introduced a 3-form

$$E = 2i\bar{p} \sum_k \xi_k \bar{\zeta}_k + s \sum_k (\xi_k \bar{\xi}_k - \zeta_k \bar{\zeta}_k), \quad (27)$$

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<sup>3</sup>See Appendix B for more detail.

which has spin  $+1/2$  under  $SU(2)_J$ . Then,  $\partial\Psi$  is written as a polynomial of  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$ , and the harmonicity condition  $\partial\Psi = 0$  means that the coefficients of independent monomials should all vanish. The resulting equations form a set of first order differential equations for the coefficient functions  $F_a(\rho)$  in (20) and many additional algebraic relations. In addition, from the self-duality condition of  $\Psi$  we will have further algebraic relations. Solving directly all these differential and algebraic equations, in principle, one can obtain the middle form, though it would not be a simple task in general  $4n + 4$  dimensions.

Given the uniqueness of the solution, however, the actual route we take is to work with a plausible ansatz which is chosen after some experience with a few low dimensional cases up to, say, 20 dimensions ( $n = 4$ ). Remarkably, it turns out that the coefficient functions of the middle form are not independent but may be expressed in terms of two functions in the following way,

$$\Psi = F_n(\rho) \sum_l (a_l A^2 + b_l B^2 + c_l C) D^l B^{n-2l-1} + G_n(\rho) \sum_l d_l A D^l B^{n-2l}, \quad (28)$$

where

$$\begin{aligned} F_n(\rho) &= \frac{1}{\rho^2(\rho^2 + 1)^{n+2}}, \\ G_n(\rho) &= \frac{1}{\rho^4(\rho^2 + 1)^n} + nF_n(\rho), \end{aligned} \quad (29)$$

and  $a_l$ ,  $b_l$ ,  $c_l$  and  $d_l$ 's are numerical coefficients to be determined from the conditions of harmonicity and self-duality for  $\Psi$ . Here,  $b_0$  may be put to be zero using the identity (23). Inserting this ansatz into  $\partial\Psi = 0$  and using (26), we obtain many differential and algebraic consistency equations for  $F_n(\rho)$ ,  $G_n(\rho)$  and other coefficients. Eventually, with the above form of  $F_n(\rho)$  and  $G_n(\rho)$ , all the differential equations reduce to algebraic ones which have a unique solution up to an overall normalization,

$$\begin{aligned} a_l &= (n - 2l)(n - 2l + 1)U_l, \\ b_l &= -2lU_l, \\ c_l &= a_l, \\ d_l &= (n - 2l + 1)U_l, \end{aligned} \quad (30)$$

where

$$U_l = (-1)^l \binom{n-l+1}{l}. \quad (31)$$

Note that these coefficients are similar to those in Eq. (23) which is crucially used in the calculation. In addition, it can be shown that the resulting  $\Psi$  is indeed self-dual, the proof of which is given in Appendix B. This solution is normalizable as we see from the form of  $F_n(\rho)$  and  $G_n(\rho)$ . Also, for  $n \leq 2$ , it reduces to the corresponding  $L^2$  harmonic form found in [6].

## 4 The Ground State of Instanton and Free Motion of Fivebranes

The  $L^2$  harmonic form of previous section finds an obvious interpretation here in the context of DLCQ of fivebrane theory. One set of states on fivebranes that separates out from the rest is the free center of mass degrees of freedom which consist of 5 scalars, 4 symplectic-majorana spinors, and one chiral tensor multiplet. This multiplet is composed of 16 degrees of freedom. On the other hand, an  $L^2$  ground state of instanton comes with additional degeneracy due to the superpartner of free, so far neglected,  $R^4$  part of moduli dynamics. There are 4 free complex fermions, which induces degeneracy of 16, exactly the right amount to form the tensor multiplet.<sup>4</sup>

A consistency check of this interpretation can be found when we separate fivebranes from each other. The counting of the normalizable ground state in fact changes dramatically in this case. This induces a potential to the moduli space dynamics and the counting of the ground state becomes a good deal easier. For  $m = n + 2$  separated fivebranes, the number of  $L^2$  ground states is precisely  $m$  [3, 31]. On the other hand, in this phase, each and every fivebrane is described by a free tensor multiplet of its own. Thus, when one performs DLCQ of this background, one must find KK tower of each and every one of these separated fivebranes. The emergence of extra  $L^2$  ground states is thus precisely what we need to attribute them to free tensor multiplet.<sup>5</sup>

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<sup>4</sup>An interesting question is how other states from interacting part of fivebrane theory, namely interacting (2,0) theory. Aharony et.al [18] argued that the relevant quantity is the compact cohomology of the total instanton moduli space. Translated to compact cohomology of the relative part of the instanton moduli space, the statement becomes

$$H_{compact}^{2n+2l} = Z, \quad (32)$$

for  $l = 1, \dots, n + 2$ . With the single exception of  $l = 1$  case which we already interpreted as coming from free part of (2,0) theory, it is not clear if any of these cohomology generators can be represented by an harmonic form.

<sup>5</sup>Another interesting question arises here. Whether and how these extra  $m - 1 = n + 1$  states might be related to the extra states,  $H_{compact}^{2n+2l} = Z$  for  $l = 2, \dots, n + 2$ , when the fivebranes are all

## 5 Quantum Screening of Non-Abelian Monopoles

As mentioned in Introduction, Calabi manifold also makes an appearance in the context of non-Abelian monopoles [2]. Monopoles in question are those that arise when  $SU(n+4)$  is broken to  $U(1) \times SU(n+2) \times U(1)$  and carry either of the  $U(1)$  charges as well as non-Abelian  $SU(n+2)$  charges individually. One could write down family of solution involving one of each, whose combined asymptotic field has no  $SU(n+2)$  charge. The low energy dynamics of such a soliton pair is described by 4 center of mass, thus free, degrees of freedom and additional  $4(n+2)$  interacting ones. The metric for such moduli space has been derived in Ref. [2], which we present below.

### 5.1 Moduli Space of Non-Abelian Monopoles

Consider  $SU(n+4)$  spontaneously broken to  $U(1) \times SU(n+2) \times U(1)$ . Of  $n+3$  possible fundamental monopoles, all but the first and the last would become massless, whose degrees of freedom is known to appear in the relative low energy interaction of the two massive monopoles. Two massive monopoles is charged with respect to each  $U(1)$ , and both are charged under the  $SU(n+2)$ .

The relative moduli space of the monopoles has the topology of  $R^{4(n+2)}$ . With the reduced mass of the two massive monopoles,

$$\bar{\mu} \equiv \frac{m_1 m_{n+3}}{m_1 + m_{n+3}}, \quad (33)$$

the metric is [2]

$$\mathcal{G}_{\text{rel}} = \frac{g^2}{8\pi} \mathcal{G}_0 + \bar{\mu} \left( \sum_A d\vec{x}_A \right)^2 - \frac{g^2 \bar{\mu}}{g^2 + 8\pi \bar{\mu} \sum_B x_B} \left( \sum_A x_A (d\psi_A + \cos \theta_A d\phi_A) \right)^2, \quad (34)$$

where

$$\mathcal{G}_0 = \sum_A \frac{1}{x_A} d\vec{x}_A^2 + x_A (d\psi_A + \cos \theta_A d\phi_A)^2 \quad (35)$$

is a flat  $R^{4n+8}$  metric. The summations are over  $A = 1, \dots, n+2$ . The metric is hyper-Kähler, as it must be, and the three independent Kähler forms are

$$w^{(a)} = \frac{g^2}{8\pi} w_0^{(a)} - \frac{\bar{\mu}}{2} \epsilon^{abc} \left( \sum_A dx_A^b \right) \wedge \left( \sum_B dx_B^c \right). \quad (36)$$

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coincident.

The magnetic coupling constant  $g$  is related to the electric coupling constant  $e$  by  $eg = 4\pi$ . This space has  $SU(n+2)$  triholomorphic Killing vector fields, which comes from the unbroken  $SU(n+2)$  gauge symmetry.

## 5.2 Classical Potential

When there is more than one adjoint Higgs vev turned on, the low energy dynamics acquires a new set of terms involving static potential. It has the general form [32, 33, 34],

$$\mathcal{V} = \frac{1}{2}(G_\mu G^\mu + \nabla_\mu G_\nu \bar{\lambda}^\mu \sigma_3 \lambda^\nu), \quad (37)$$

where  $G$  is a linear combination of triholomorphic Killing vector fields associated with unbroken  $U(1)$ 's and  $\lambda$ 's are certain two-component fermionic collective coordinates. In the above metric, the only such  $U(1)$  Killing vector available for  $G$  is

$$\sum_A \frac{\partial}{\partial \psi^A}. \quad (38)$$

The bosonic potential is determined by the size  $a$  of the second Higgs vev, and behaves as

$$\mathcal{C}a^2 \left( \sum_A x_A - \frac{8\pi\mu}{g^2 + 8\pi\bar{\mu}\sum_B x_B} \left( \sum_A x_A \right)^2 \right) = \mathcal{C}a^2 \frac{g^2 L}{g^2 + 8\pi\mu L}, \quad (39)$$

with the constant  $\mathcal{C}$  to be determined and  $L \equiv (\sum_B x_B)$ .

When the monopoles are separated at large distance, the nontrivial part of potential behaves as  $\sim 1/L$ . Expanding the potential for large  $L$

$$\mathcal{V}_B = \mathcal{C}a^2 \times \frac{g^2}{8\pi\bar{\mu}} \times \frac{1}{1 + g^2/8\pi\bar{\mu}L} \simeq \frac{\mathcal{C}}{4\pi} \times \left( \frac{g^2 a^2}{2\bar{\mu}} - \frac{g^2}{16\pi} \frac{g^2 a^2}{\bar{\mu}^2} \frac{1}{L} + \dots \right). \quad (40)$$

BPS mass formulae of the monopoles fixes the value of constant,  $\mathcal{C} = 4\pi$ , and we find the following asymptotic form of potential,

$$\mathcal{V}_B = \frac{g^2 a^2}{2\bar{\mu}} - \frac{g^2}{8\pi} \frac{g^2 a^2}{2\bar{\mu}^2} \frac{1}{L} + \dots \quad (41)$$

Given a fixed separation of the two massive monopoles,  $\vec{R} = \sum_A \vec{x}_A$ , the minimum value of  $L = \sum_A |\vec{x}_A|$  is  $R = |\vec{R}|$ . Thus, classically, there is an attractive potential between the two massive monopoles whose asymptotic form goes as

$$(\mathcal{V}_B)|_{\text{minimum}} = \frac{g^2 a^2}{2\bar{\mu}} - \frac{g^2}{8\pi} \frac{g^2 a^2}{2\bar{\mu}^2} \frac{1}{R} + \dots \quad (42)$$

One can think of this “massless” monopoles settling down at their classical vacuum, corresponding to being lined up along the straight line between the two massive monopoles. Quantum mechanically, on the other hand, they tends to spread out simply due to quantum fluctuations. This increases the effective value of  $L$  and soften the attractive potential. This effect is simplest to observe when massive monopoles are held at fixed locations, which we achieve by taking an infinite mass limit while keeping  $ga/\mu$  finite.

### 5.3 Infinite Mass Limit

One important difference in low energy dynamics of non-Abelian monopoles is that not all degrees of freedom are associated with translations and internal rotations of the massive cores. Rather, one finds additional non-Abelian long-range degrees of freedoms, which was dubbed as massless monopole clouds [2]. In the above example, dynamics of these massless cloud emerges when we take hyper-Kähler quotient with respect to  $U(1)$  generated by  $G$ , or equivalently take a limit where the masses of the two massive monopoles goes to infinite. The resulting metric is

$$\frac{g^2}{8\pi} \left( C_{AB} d\vec{x}_A \cdot d\vec{x}_B + (C^{-1})_{AB} (d\psi_A + \vec{\omega}_{AC} \cdot d\vec{x}_C) (d\psi_B + \vec{\omega}_{BD} \cdot d\vec{x}_D) \right). \quad (43)$$

where now  $A$  runs from 1 to  $n+1$  (instead of  $n+2$ ). The periodic coordinate  $\psi_A$  has period,  $4\pi$ , and the symmetric matrix  $C_{AB}$  has the form,

$$C_{AB} = \left( \frac{\delta_{AB}}{|\vec{x}_A|} + \frac{1}{|\sum_{E=1}^{n+1} \vec{x}_E - \vec{R}|} \right). \quad (44)$$

The last term is common to all components. The vector potentials  $\vec{\omega}_{AB}$  are again related to  $C_{AB}$  by

$$\vec{\nabla}_D C_{AB} = \vec{\nabla}_D \times \vec{\omega}_{AB}. \quad (45)$$

This metric has the same form as in (1), so the relevant moduli space here is again Calabi manifold.

Furthermore, the quantity  $L$  that appears in the classical bosonic potential is now  $\sum_{A=1}^{n+1} x_A + |\sum_{A=1}^{n+1} \vec{x}_A - \vec{R}|$ , which was previously identified with the coordinate  $\rho^2 R$  in Section 3. Dynamics of massless monopoles thus inherit the potential of the whole monopole dynamics, whose bosonic part is,

$$\mathcal{V}_B = \left( \frac{g^2 a^2}{2\bar{\mu}} - \frac{g^2}{8\pi} \frac{g^2 a^2}{2\bar{\mu}^2} \frac{1}{R\rho^2} + \cdots \right). \quad (46)$$



When this potential is deemed to be small, one can treat it as a perturbation over the purely kinetic dynamics on the Calabi manifold. In particular, the quantum effective potential for the pair of massive monopoles at separation  $\vec{R}$  would be given by

$$\langle \Psi | \mathcal{V} | \Psi \rangle / \langle \Psi | \Psi \rangle, \quad (47)$$

where  $\Psi$  is the normalizable ground state of the massless monopoles in the absence of the potential.

## 5.4 Quantum Corrected Potential

We would like to estimate the quantum corrected effective potential between two massive monopoles discussed earlier. In the limit where the two massive monopoles are infinitely heavy and separated by a fixed distance  $L$ , the Hamiltonian is just that of the cloud and is governed by Calabi manifold of scale  $\sqrt{R}$ . The Hamiltonian is decomposed into two parts,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V}, \quad (48)$$

where  $\mathcal{V}$  is the supersymmetric potential of the original dynamics appropriately reduced. Both term has an implicit dependence on the distance  $R$ .

The harmonic form of the previous section is in effect the ground state wavefunction with respect to  $\mathcal{H}_0$ . The quantum corrected potential is obtained via standard perturbation theory,

$$V_{eff}(R) = \langle \Psi | \mathcal{V} | \Psi \rangle / \langle \Psi | \Psi \rangle. \quad (49)$$

This is further simplified by noticing that terms involving fermions in  $\mathcal{V}$  always change the fermion number by two. On the other hand, the fermion number is really the degree of the wavefunction expressed as differential form, and thus any such operator will have vanishing expectation on any middle form. Thus, the effective potential is found by evaluating expectation of purely bosonic part

$$V_{eff}(R) = \langle \Psi | \mathcal{V}_B | \Psi \rangle / \langle \Psi | \Psi \rangle, \quad (50)$$

For evaluation of this, a useful identity is

$$*(\Psi \wedge * \Psi) = *(\Psi \wedge \Psi) = k_n \times \left( n(n+4)F_n(\rho)^2 + G_n(\rho)^2 \right), \quad (51)$$

for some numbers  $k_n$ . Then, we have

$$V_{eff}(R) = \langle \Psi | \mathcal{V}_B | \Psi \rangle / \langle \Psi | \Psi \rangle = \frac{g^2 a^2}{2\bar{\mu}} - \frac{g^2}{8\pi} \frac{g^2 a^2}{2\bar{\mu}^2} \frac{S(n)}{R} + \dots, \quad (52)$$

where

$$S(n) \equiv \left( \int \left( n(n+4)F_n(\rho)^2/\rho^2 + G_n(\rho)^2/\rho^2 \right) \right) / \left( \int \left( n(n+4)F_n(\rho)^2 + G_n(\rho)^2 \right) \right). \quad (53)$$

where the integrals are over the Calabi manifold. For large  $n$ , this expression approaches zero as,

$$S(n) \simeq \frac{5}{3n}. \quad (54)$$

Thus we find that quantum effect tends to screen the leading attractive interaction  $\sim 1/R$  down to  $\sim 1/(nR)$  for large  $n$ . This is reminiscent of the screening effect found by Maldacena [23] and by Rey and Yee [22], from AdS/CFT picture of Wilson line for quark-anti-quark pair. The precise behavior of the latter effect does not match up with the above, which may be attributed to the fact that we are truncating most of massless non-Abelian degrees of freedom except those associated with massless monopoles. It is nevertheless interesting that a strong screening effect appears already at the level of the low energy approximation for monopoles.

## 6 Summary

We considered a supersymmetric sigma model onto Calabi manifold of arbitrary dimensions, and found an exact, self-dual, square-normalizable, ground state of the quantum mechanics. This wavefunction is used to show that a non-commutative instanton of Super Yang-Mills theory in 4+1 dimensions has a finite sized quantum ground state, and is interpreted as the first KK mode of DLCQ description of coincident fivebrane theories. Also the same wavefunction demonstrates how interaction between non-Abelian monopoles in partially broken Yang-Mills theories experience screening effect at long range.

The Calabi manifold should play a further role in understanding (2,0) theories. Physics we probed here are relevant to the free part of fivebrane dynamics, and interacting (2,0) theory is not really addressed. In the spirit of DLCQ approach [35], one should be able to discover some physics of (2,0) theories from supersymmetric quantum mechanics on the Calabi manifold. One interesting and immediate question is, for instance, whether the topological information such as axial and gravitational anomaly [15] can be computed in such an approach.

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## Appendix A

Here we summarize the exterior algebra for left-invariant one-forms obtained by Cvetic et.al. [6] which is needed to derive (26). Let  $L_A^B$  be the left-invariant one-forms of  $SU(n+2)$ , where  $L_A^B$  and  $(L_A^B)^\dagger = L_B^A$ , which satisfy the exterior algebra

$$dL_A^B = iL_A^C L_C^B. \quad (55)$$

Splitting the  $SU(n+2)$  index  $A$  as  $A = (1, 2, a)$ , one can then identify  $4n+3$  generators of the coset  $SU(n+2)/U(n)$  as a real one-form

$$\lambda = L_1^1 - L_2^2, \quad (56)$$

and complex one-forms

$$\sigma_a = L_1^a, \quad \Sigma_a = L_2^a, \quad \nu = L_1^2, \quad (57)$$

and their complex conjugates. For these, the exterior algebra reduces to

$$\begin{aligned} d\sigma_a &= \frac{i}{2}\lambda\sigma_a + i\nu\Sigma_a + \cdots, \\ d\Sigma_a &= -\frac{i}{2}\lambda\sigma_a + i\bar{\nu}\Sigma_a + \cdots, \\ d\nu &= i\lambda\nu + i\sigma_a\bar{\Sigma}_a, \\ d\lambda &= 2i\nu\bar{\nu} + i\sigma_a\bar{\sigma}_a - i\Sigma_a\bar{\Sigma}_a, \end{aligned} \quad (58)$$

where  $\cdots$  represents the terms lying outside the coset. Noting that  $s$ ,  $p$ ,  $\sigma_a$  and  $\bar{\Sigma}_a$  are holomorphic one-forms, it is straightforward to obtain (26) from the above equations.

## Appendix B

The identity in Eq. (23) can be obtained in the following way: Add an additional pair of doublets, so that we have

$$\begin{pmatrix} \xi_k \\ i\zeta_k \end{pmatrix}, \quad k = 1, \dots, n+1, \quad (59)$$

and

$$\begin{pmatrix} i\bar{\zeta}_k \\ \bar{\xi}_k \end{pmatrix}, \quad k = 1, \dots, n+1. \quad (60)$$

Furthermore, we extend  $U(n)$  to an  $U(n+1)$  acting on them as fundamental and anti-fundamental, respectively. Starting with the above, we may build a pair of spin  $(n+1)/2$  representations (under  $SU(2)_J$ ) by contracting the  $n+1$  doublets in the fundamental of  $U(n+1)$  with a completely anti-symmetric tensor  $\epsilon_{123\dots(n+1)}$ , and similarly for those in the anti-fundamental.

We then build a singlet by multiplying the two spin  $(n+1)/2$  representations. Finally we may convert the expression with two  $\epsilon$ 's into an expression with  $(n+1)$  inner products under  $U(n+1)$ , thereby arriving at the right hand side of Eq. (23) with  $B$  and  $D$  replaced by the same expressions but now with the sums over  $k = 1, \dots, n+1$ . Call them  $\tilde{B}$  and  $\tilde{D}$ , respectively. When we take the hypothetical  $(n+1)$ -th doublets to zero,  $\tilde{B}$  reduces to  $B$  and  $\tilde{D}$  reduces to  $D$ . On the other hand, this makes each of the two spin  $(n+1)/2$  quantities vanish identically since each has to involve a factor of the  $(n+1)$ -th doublets. The spin singlet built from them should vanish as a result, and thus the expression on the right hand side of Eq. (23) has to vanish by itself when the sums are taken over  $k = 1, \dots, n$ .

Obviously more such identities can be obtained for invariant  $2n+2l$  forms with  $l > 1$ , but these additional identities are irrelevant for our purpose.

## Appendix C

In this appendix we show that the middle form found in the main part is self-dual. First we observe that

$$A(\sum_k \xi_k \bar{\xi}_k)^l (\sum_k \zeta_k \bar{\zeta}_k)^m (\sum_k \xi_k \bar{\zeta}_k \sum_k \bar{\xi}_k \zeta_k)^q, \quad l + m + 2q = n, \quad (61)$$

is dual to

$$A(\sum_k \xi_k \bar{\xi}_k)^m (\sum_k \zeta_k \bar{\zeta}_k)^l (\sum_k \xi_k \bar{\zeta}_k \sum_k \bar{\xi}_k \zeta_k)^q, \quad (62)$$

since the last factor which containing cross terms of  $\xi$ 's and  $\zeta$ 's cannot be generated by  $\xi\bar{\xi}$  or  $\zeta\bar{\zeta}$  combinations. This can be verified by explicit calculations using Levi-Civita  $\epsilon$  tensors and combinatorics. Then it is easy to see that the invariant middle form

$$AD^l B^{n-2l}, \quad l = 0, 1, \dots, [n/2], \quad (63)$$

is self-dual since it is expanded in terms of the above forms in a symmetric way with respect to  $\xi$  and  $\zeta$ .

As far as hodge duality is concerned, we can treat the 1-forms  $s$  and  $-ip$  on equal footing with  $\xi$ 's and  $\zeta$ 's, which leads us to consider the quantities

$$\begin{aligned} \tilde{B} &= \sum_k (\xi_k \bar{\xi}_k + \zeta_k \bar{\zeta}_k) + s\bar{s} - p\bar{p} = B + A, \\ \tilde{D} &= -(\sum_k \xi_k \bar{\zeta}_k - isp)(\sum_k \zeta_k \bar{\xi}_k + i\bar{s}\bar{p}) + (\sum_k \xi_k \bar{\xi}_k + s\bar{s})(\sum_k \zeta_k \bar{\zeta}_k - p\bar{p}) \\ &= D + A^2 + (AB - C)/2. \end{aligned} \quad (64)$$

Equation (63) implies that  $D^l B^{n-2l}$  is self-dual in  $4n$  dimensions spanned by  $\xi$ 's and  $\zeta$ 's. In  $4n+4$  dimensions, it means that  $\tilde{D}^l \tilde{B}^{n-2l+1}$  is self-dual. Expanding it, we find

$$\begin{aligned} \tilde{D}^l \tilde{B}^{n-2l+1} &= D^l B^{n-2l+1} \\ &+ (n-2l+1)AD^l B^{n-2l} + \frac{l}{2}AD^{l-1}B^{n-2l+2} \\ &+ -\frac{l}{2}CD^{l-1}B^{n-2l+1} \\ &+ \frac{l(l+1)}{2}A^2D^{l-1}B^{n-2l+1} + \frac{l}{2}(n-2l+1)A^2D^{l-1}B^{n-2l+1} \\ &+ \frac{1}{2}(n-2l)(n-2l+1)A^2D^l B^{n-2l-1}. \end{aligned} \quad (65)$$

Since the terms in the second line is self-dual, we see that the term linear in  $C$  should be self-dual and the term in the first line is dual to the terms quadratic in  $A$ . With these dual relations, it is easy to check that the middle form  $\Psi$  is indeed self-dual with the coefficients in (30).

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